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SUMMATION OF ELLIPSOIDS IN THE GUARANTEED ESTIMATION PROBLEM*

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The problem of the guaranteed estimation of the phase space of a linear dynamic system when there are independent external disturbances of bounded magnitude is considered. Some methods of approximate guaranteed estimation by ellipsoids were developed in /1-4/. In this paper, we construct an approximation by an ellipsoid which is optimal in the sense of a general performance criterion - the Minkowski sum of ellipsoids. Some basic properties of the operation are indicated. The results are applied to estimate the reachability region of a multi-input linear controlled system.

1. *The basic problem.*

We denote by $E(a, Q)$ the ellipsoid $\{x;\ (Q^{-1}(x-a),\ x-a) \leqslant 1\}$. Here x is an n-dimensional vector, a is the n-dimensional vector of the centre of the ellipsoid, Q is an $n\times n$ symmetric positive definite matrix, and (a, b) is the scalar product of the vectors a and b.

An equivalent definition of the ellipsoid may be written in the form

$$E(a, Q) = \{x;\ (x, y) \leqslant (a, y) + (Qy, y)^{1/2},\ \forall y \in \mathbf{R}^n\} \tag{1.1}$$

which indicates that the supporting function of the ellipsoidal set $H_{E(a,Q)}(y) = \sup_{x\in E(a,Q)}(x, y)$ is defined by the formula $H_{E(a,Q)}(y) = (a, y) + (Qy, y)^{1/2}$ (see, e.g., /3/).

Formula (1.1) enables us to extend the definition of $E(a, Q)$ to the degenerate case, when some of the ellipsoid axes are of length zero, which corresponds to a singular matrix Q.

Henceforth, the indices i, j take the values $1, 2, \ldots, m$; summation over these indices is from 1 to m.

Consider the Minkowski sum of m ellipsoids,

$$\Omega = \{x = \textstyle\sum x_i;\ x_i \in E_i = E(a_i, Q_i)\}$$

The condition that the ellipsoid $E(a, Q)$ includes the region Ω can be written as

$$(a, y) + (Qy, y)^{1/2} \geqslant \textstyle\sum [(a_i, y) + (Q_iy, y)^{1/2}],\quad \forall y \in \mathbf{R}^n \tag{1.2}$$

Here we have used the fact that the supporting function of the sum of non-empty convex sets is equal to the sum of the supporting functions of these sets and that the inclusion $D_1 \subset D_2$ of closed convex sets D_1 and D_2 is equivalent to the inequality $H_{D1} \leqslant H_{D2}$ (see, e.g., /5/).

The inequality (1.2) holds, in particular, for $y' = -y$. Adding the inequalities (1.2) for y and y', we obtain a constraint on the matrix Q alone:

$$(Qy, y)^{1/2} \geqslant \textstyle\sum (Q_iy, y)^{1/2},\quad \forall y \in \mathbf{R}^n \tag{1.3}$$

Clearly, if inequality (1.3) holds and $a = \Sigma a_i$, then inequality (1.2) also holds.

The matrix Q satisfying (1.3) is chosen so that

$$L(Q) \to \min \tag{1.4}$$

Here $L(Q)$ is a smooth monotone increasing function of the matrix Q; i.e. if $Q_1 \geqslant Q_2$ (which means that $(Q_1\eta, \eta) \geqslant (Q_2\eta, \eta), \forall \eta \in \mathbf{R}^n$ or, equivalently, $E(0, Q_1) \supset E(0, Q_2)$), then $L(Q_1) \geqslant L(Q_2)$. The monotonicity condition can be restated in the form $\partial L/\partial Q \geqslant 0$, where $\partial L/\partial Q$ is the gradient of the function L.

Note that the centre of the ellipsoid E required is equal /2/ to the sum of the centres of the ellipsoids E_i. A quadratic form of the vector y may be written as a linear form of a matrix X of rank 1,

$$(Qy, y) = \operatorname{tr}(QX),\quad X = y^*y$$

where the asterisk denotes transposition. Let $\langle Q, X\rangle = \operatorname{tr}(Q, X)$. Then (1.3) is rewritten

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$$\langle Q, X\rangle^{1/2} \geqslant \Sigma \langle Q_i, X\rangle^{1/2} \qquad (1.5)$$

or, after squaring,

$$\langle Q, X\rangle \geqslant \sum_{i,j} \langle Q_i, X\rangle^{1/2} \langle Q_j, X\rangle^{1/2}, \quad X = y^*y, \quad \forall y \in \mathbf{R}^n \qquad (1.6)$$

since both sides of the inequality (1.5) are non-negative numbers. The function on the right-hand side of inequality (1.6) is concave on the set of non-negative definite matrices (this set is the convex hull of the matrices of rank 10 $X = y^*y$, $y \in \mathbf{R}^n$).

We will show that the extension of (1.6) to the set of all non-negative definite matrices does not introduce additional constraints in problem (1.3), (1.4).

We will first demonstrate this for a simple problem with the constraint

$$\langle Q, X\rangle \geqslant \Phi(X) = \langle A, X\rangle^{1/2} \langle B, X\rangle^{1/2}, \quad X = y^*y, \quad \forall y \in \mathbf{R}^n \qquad (1.7)$$

where A and B are non-negative definite matrices.

Let $p : W \to \mathbf{R}^2$ be a linear mapping such that $p_1 = \langle A, X\rangle$, $p_2 = \langle B, X\rangle$, where W is the set of non-negative definite matrices.

Lemma 1. The sets W and W_1 of the matrices of rank 1 y^*y, $y \in \mathbf{R}^n$, have the same image under the mapping p.

Proof. Without loss of generality, assume that A and B are diagonal matrices, $A = \operatorname{diag}\{a^1, \ldots, a^n\}$, $B = \operatorname{diag}\{b^1, \ldots, b^n\}$. Then

$$p_1 = \sum a^k X_{kk}, \quad p_2 = \sum b^k X_{kk}, \quad \forall X \in W$$

Here and in the example at the end of Sect.1 the index k takes the values $1, 2, \ldots, n$; the summation over this index is from 1 to n.

Note that the same point (p_1, p_2) can be obtained also as the image of the matrix rank 1 y^*y, where $y^k = \sqrt{X_{kk}}$. The lemma is proved.

It is essential that the function $\Phi(X)$ is representable as a superposition: $\Phi = \varphi \circ p$, where $\varphi : \mathbf{R}^2 \to \mathbf{R}$ is defined by the formula $\varphi(p_1, p_2) = p_1^{1/2} p_2^{1/2}$. Also note that the function φ is defined on some cone in the positive orthant of the plane $\mathbf{R}^2$ (in particular, when $B = \alpha A$, $\alpha > 0$, this is a ray issuing from the origin).

The problem of extending (1.7) to the set W without introducing additional constraints involves finding a function Φ' - the minimal concave function on the set W that coincides with Φ on W_1. This problem is solved by the following lemma.

Lemma 2.

$$\Phi'(X) = \Phi(X), \quad \forall X \in W.$$

Proof. Since any matrix from W is representable as a finite linear combination of matrices from W_1, then by the definition of Φ', we have

$$\Phi'(X) = \inf_{X = \Sigma y_l^* y_l} \sum \Phi(y_l^* y_l) \geqslant \inf_{p(X) = \Sigma p(y_l^* y_l)} \sum \varphi(p(y_l^* y_l)) = \varphi'(p(X)) = \varphi(p(X)) = \Phi(X)$$

Here and in Lemma 3, the index l is the number of the element in the linear combination. Summation is over all the elements of the representation.

The last equality follows from Lemma 1. Thus $\Phi'(X) \geqslant \Phi(X)$. But, from the definition of Φ', $\Phi'(X) \leqslant \Phi(X)$. The lemma is proved.

From Lemma 2 we obtain that inequality (1.7) is equivalent to the following inequality:

$$\langle Q, X\rangle \geqslant \langle A, X\rangle^{1/2} \langle B, X\rangle^{1/2}, \ \forall X \in W$$

We will prove a similar proposition for the constraint (1.6). This will require the following lemma.

Lemma 3.

$$\Big(\sum_k \Phi_k(X)\Big)' = \sum_k \Phi_k'(X), \quad \forall X \in W$$

Here an arbitrary finite sum over k is implied.

Proof. By definition,

$$\Big(\sum_k \Phi_k(X)\Big)' = \inf_{X = \Sigma y_l^* y_l} \sum_{k,l} \Phi_k(y_l^* y_l) \geqslant \sum_k \inf_{X = \Sigma y_l^* y_l} \sum_l \Phi_k(y_l^* y_l) = \sum_k \Phi_k'(X)$$

Note that from the definition of the operation $\Phi \to \Phi'$ we have the relationship

$$\left(\sum_k \Phi_k(X)\right)' \leqslant \sum_k \Phi_k(X) = \sum_k \Phi_k'(X)$$

The last equality follows from Lemma 2. The lemma is proved.

We have thus justified the extension of inequality (1.6) to the set W, i.e., the constraint (1.6) is equivalent to the inequality

$$\langle Q, X\rangle \geqslant \sum_{i,j} \langle Q_i, X\rangle^{1/2} \langle Q_j, X\rangle^{1/2}, \quad \forall X \in W \tag{1.8}$$

Note that inequality (1.8) defines the function $R_D(X) = \inf_{Q\in D} \langle Q, X\rangle$, which is connected with the supporting function $H_D(X)$ of feasible matrices Q (i.e., matrices for which the corresponding ellipsoids include Ω) by the relationship $R_D(X) = -H_D(-X)$. Specifically,

$$R_D(X) = \begin{cases} \sum_{i,j} \langle Q_i, X\rangle^{1/2} \langle Q_j, X\rangle^{1/2}, & X \in W \\ -\infty & , \ X \notin W \end{cases}$$

Given $R_D(X)$ we can parametrize the points Q of the boundary D minimizing the scalar product $\langle Q, X\rangle$:

$$Q = \partial R_D / \partial X \tag{1.9}$$

Formula (1.9) solves problem (1.3), (1.4) for the linear functional

$$L(Q) = \langle C, Q\rangle \tag{1.10}$$

We have thus proved the following theorem.

Theorem 1. The solution of problem (1.3), (1.4) is given by the formula

$$Q = \sum_{i,j} Q_i \frac{\langle Q_j, C\rangle^{1/2}}{\langle Q_i, C\rangle^{1/2}} \tag{1.11}$$

Denote by $E_1 \boxplus E_2 \boxplus \ldots \boxplus E_N$ the ellipsoid E minimizing the performance criterion (1.10) and containing the Minkowski sum of the ellipsoids E_p, $p = 1, 2, \ldots, N$. We will also use the notation $E = \boxplus \sum_{p=1}^{N} E_p$.

Let us state the main properties of this operation.

Property 1 (commutativity):

$$E(a_1, Q_1) \boxplus E(a_2, Q_2) = E(a_2, Q_2) \boxplus E(a_1, Q_1)$$

Property 2 (associativity):

$$\left[\boxplus \sum_{p=1}^{N} E(a_p, Q_p)\right] \boxplus E(a_{N+1}, Q_{N+1}) = \boxplus \sum_{p=1}^{N+1} E(a_p, Q_p)$$

Property 3 (similar ellipsoids):

$$E(a_1, \alpha_1^2 Q) \boxplus E(a_2, \alpha_2^2 Q) = E(a_1 + a_2, (\alpha_1 + \alpha_2)^2 Q), \ \forall \alpha_1 > 0, \alpha_2 > 0$$

Property 1 is obvious, properties 2 and 3 can be verified by substitution into (1.11). For an arbitrary performance criterion (1.4), we have the following theorem.

Theorem 2. The solution of problem (1.3), (1.4) is given by the formula

$$Q = \sum_{i,j} Q_i \frac{\langle Q_j, \partial L/\partial Q\rangle}{\langle Q_i, \partial L/\partial Q\rangle} \tag{1.12}$$

Note that for a non-linear performance criterion formula (1.12) is not a direct definition of the matrix Q, because on the right-hand side of (1.12) the gradient $\partial L/\partial Q$ is evaluated at the point Q. Therefore, formula (1.12) is an equation that has to be solved for Q. In the special case $L(Q) = \mathrm{vol}\, E(a, Q)$, the solution of (1.12) for $m = 2$ reduces to solving the two algebraic equations derived in /2/. Associativity in general does not hold here, but properties 1 and 3 are preserved.

Consider an example of approximating the minimal trace of a right parallelepiped by an ellipsoid E with the matrix Q. Note that the parallelepiped is the sum of segments representing degenerate ellipsoids, each with a single non-zero axis:

$$E_i = E(0, Q_i), \quad Q_i = \mathrm{diag}\{d_i^1, \ldots, d_i^n\}$$
$$d_i^k = \alpha_i^2 \delta_{ik}, \quad \alpha_i > 0$$

where δ_{ik} is the Kronecker delta.

For this case, (1.11) gives

$$Q = \mathrm{diag}\{d^1, \ldots, d^n\}, \quad d^k = \alpha_i \left(\sum \alpha_k\right)$$

Note that in the special case when the approximating ellipsoid is optimal by the linear performance criterion (1.10) (e.g., the trace of the ellipsoid matrix - the sum of squares of the axis half-lengths), the results are fully consistent with the corresponding operations for normal random variables.

2. *Estimation of the reachability region.* Let us apply our results in the problem of the guaranteed estimation of the state of a linear dynamic system when there are several independent external disturbances

$$x^{\cdot} = A(t)x + \sum u_i(t) \tag{2.1}$$

Here t is the time, x is an n-dimensional phase vector, and $u_i(t)$ is the n-dimensional vector of controls or disturbance functions satisfying the constraints

$$u_i(t) \in E(g_i(t), G_i(t)) \tag{2.2}$$

where $G_i(t)$ are continuous non-negative definite matrices, and $g_i(t)$ are continuous vector functions. Let the initial vector be localized in an ellipsoid:

$$x(t_0) \in E(a_0, Q_0) \tag{2.3}$$

Using the finite-difference derivation of the evolution equations of the estimating ellipsoid from /2/ and applying the new operation of ellipsoid summation, we obtain differential equations describing the motion of the optimal ellipsoids:

$$\begin{gathered} a^{\cdot} = A(t)a + \sum g_i(t) \\ Q^{\cdot} = A(t)Q + QA^*(t) + \left(\sum q_i\right) Q + \sum \frac{G_i(t)}{q_i} \\ a(t_0) = a_0, \; Q(t_0) = Q_0 \end{gathered} \tag{2.4}$$

where

$$q_i = \frac{\mathrm{tr}(G_i(t)\, \partial L/\partial Q)}{\mathrm{tr}(Q \partial L/\partial Q)} \tag{2.5}$$

The resulting ellipsoid solves the following extremal problem:

$$dL(Q)/dt \to \min \tag{2.6}$$

If we need ellipsoidal estimates that are optimal in the sense of the rate of change of the linear performance criterion

$$d\,\mathrm{tr}(CQ)/dt \to \min$$

then the solution of Eqs.(3.5), (3.6) is the same as for problem (2.1)-(2.3) with one $(m = 1)$ disturbance bounded by an ellipsoid with the parameters

$$g(t) = \sum g_i(t)$$
$$G(t) = \sum_{i,j} G_i(t) \frac{\mathrm{tr}(G_j(t)C)}{\mathrm{tr}(G_i(t)C)}$$

The ellipsoid $E(g(t), G(t))$ was obtained by summation of the ellipsoids $E(g_i(t), G_i(t))$ so as to ensure $\mathrm{tr}(G(t)C) \to \min$.

Note that the identity of solutions indicated above is based on the associativity property of the summation operation in the case of a linear performance criterion. For the general criterion (2.6), this identity does not hold.

3. *Example.*

Let us consider a simple example of the use of the summation operation, which makes it possible to carry out a qualitative comparison of the behaviour of the estimation ellipsoid in the guaranteed case and the dispersion ellipsoid in stochastic estimation.

Consider a linear dynamic system of the form (2.1) with $A(t) \equiv 0$, i.e.,

$$x^{\cdot} = u; \; u, x \in \mathbf{R}^n \tag{3.1}$$

The guaranteed approach assumes that the unknown control vector u and the initial system vector are contained in given sets. We define them by ellipsoids, $x(0) \in E(a_0, Q_0)$, $u \in E(0, G)$. Then clearly the exact reachability region of the given system by the time t is defined as the sum of the sets $E(a_0, Q_0)$ and $E(0, Gt^2)$.

We approximate the exact reachability region by the ellipsoid $E(a, Q)$ so that $\operatorname{tr} Q \to \min$. Hence, $E(a, Q) = E(a_0, Q_0) \boxplus E(0, Gt^2)$. Computations using (1.11) give

$$a = a_0, \; Q = Q_0 + t(\alpha Q_0 + \alpha^{-1} G) + t^2 G$$
$$\alpha = \sqrt{\operatorname{tr} G / \operatorname{tr} Q_0} \tag{3.2}$$

Note that formula (3.2) gives an exact solution of Eqs.(2.4), (2.5). This means that, for system (3.1), the locally optimal estimates are identical with estimates optimal at the terminal time.

If now u in (3.1) is interpreted as ideal white noise with intensity R (i.e., $M\{u^*(t) u(t+\tau)\} = R\delta(\tau)$, where M is the expectation operator and δ is the Dirac delta-function), and the initial vector is assumed to be a normal random variable with given mean a_0 and covariance matrix D_0, then the parameters of the dispersion ellipsoid in this problem (the expectation value a, covariance matrix D) are given by

$$a = a_0, \; D = D_0 + Rt \tag{3.3}$$

Thus, the qualitative difference between white noise and disturbances of bounded magnitude leads to different evolutionary dependences (3.2) and (3.3). In particular, for large t, the size of the estimation ellipsoid (3.2) increases approximately linearly, while the size of the dispersion ellipsoid (3.3) is approximately proportional to the square root of time.

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REFERENCES

1. KRASOVSKII N.N., Theory of Motion Control, Nauka, Moscow, 1968.
2. CHERNOUS'KO F.L., Optimal guaranteed estimates of uncertainties using ellipsoids, 1, Izv. Akad. Nauk SSSR, Tekh. Kibern., 3, 1980.
3. OVSEYEVICH A.I., Extremal properties of ellipsoids approximating reachability regions, Problemy Upravleniya i Teorii Informatsii, 12, 1, 1983.
4. KHONIN V.A., Guaranteed state estimates of linear systems using ellipsoids, In: Evolving Systems in Estimation Problems, Izd. UNts Akad. Nauk SSSR, Sverdlovsk, 1985.
5. ROCKAFELLAR R., Convex Analysis, Princeton Univ. Press, 1970.

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